

Regularization of the Reissner-Nordström black hole

S. Habib Mazharimousavi* and M. Halilsoy†

*Department of Physics, Eastern Mediterranean University,
Gazimağusa, North Cyprus, Mersin 10 - Turkey.*

(Dated:)

An inner de Sitter region is glued smoothly and consistently with an outer Reissner-Nordström (RN) spacetime on a spherical thin-shell. Mass and charge of the outer RN spacetime are defined by the de Sitter and shell parameters. Radius of the shell plays the role of a cut-off which by virtue of regular de Sitter inside removes the singularity at $r = 0$. For stability the perturbed shell must satisfy a modified polytropic equation of state.

I. INTRODUCTION

Since the inception of the cut and paste technique following the seminal work of Israel's junction conditions [1] the topic of thin-shells has been popularized extensively. Application of thin-shells to wormholes [2] in general relativity has been another major title that found vast applications. In that construction (preferably) two asymptotically flat spacetimes are glued at a minimal radius that defines the throat of the wormhole [3]. Through that throat an observer passes from one universe to the other easily. A wormhole may connect two black holes which may be interpreted in the language of modern physics as entanglement [4]. It should also be reminded that the existence of a minimum radius lead Einstein and Rosen to interpret a wormhole as a geometrical model of a particle [5]. Very special spacetimes satisfy Israel's junction conditions [1] to be glued smoothly [6–8]. In [6, 7] inner flat / Minkowski spacetime was glued to the outer extremal RN. However, Zaslavskii in [8] has shown that Minkowski spacetime can not be glued smoothly to extremal RN but instead Bertotti-Robinson spacetime was successfully glued to extremal RN black hole at its horizon.

In this letter we glue an inner de Sitter with an outer Reissner-Nordström (RN) spacetime on a spherical shell that satisfies the junction conditions of smooth match. This, however, means that no energy momentum tensor is presented on the interface hyperplane [8]. We employ this property by replacing the singular inside of a RN black hole with a regular de Sitter spacetime. The external RN spacetime which has parameters mass (M) and charge (Q) are determined from the satisfaction of the boundary conditions for a smooth match. Stated otherwise, the mass and charge are defined 'from geometry' in accordance with Wheeler's geometrodynamics [9]. Regularization is to be understood in the sense that is reminiscent of some renormalization / regularization techniques that were used in field theory. The aim in these techniques was to eliminate divergences in field theory. In doing this experimental values of particles, such as charge,

mass, magnetic moment etc. were used as guidelines. Insertion of measured quantities into the theory played major role in choosing the cut-offs. As a result finite quantities emerged from the divergent ones, as a physical requirement.

In general relativity also singularities, i.e., diverging curvature invariants lie at the heart of gravitational theory. Most black holes admit singularities at their center which make invariants divergent. The worst of such singularities is the spacelike ones as encountered in the Schwarzschild black hole. Addition of electric charge (i.e., the RN solution) makes the central singularity timelike, which is the subject matter of the present article. By cutting the central singularity and pasting a regular de Sitter spacetime we get rid of the $r = 0$ singularity. In turn, the shell must satisfy certain conditions, especially upon perturbation for stability requirement on energy-momentum of fluid arises naturally. This is in the form of a modified polytropic fluid whose energy density and transverse pressures satisfy the conservation law. The total energy plots suggest an energy zone that makes the shell and therefore our model, stable against linear radial perturbations.

II. THE MODEL

In 3 + 1–dimension, let's consider the following static, spherically symmetric spacetimes

$$ds^2 = -f_i(r_i) dt_i^2 + \frac{dr_i^2}{f_i(r_i)} + r_i^2 (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2) \quad (1)$$

for inside ($i = 1$) and outside ($i = 2$) of a timelike shell defined by $F := r - R_0 = 0$. Following the Israel junction formalism [1], the induced metric on the shell is found to be

$$ds^2 = -d\tau^2 + R_0^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (2)$$

and the energy momentum tensor components on the shell are found to be from $S_\mu^\nu = \text{diag}(\sigma_0, p_0, p_0)$

$$\sigma_0 = -\frac{1}{4\pi G} \left(\frac{\sqrt{f_2(R_0)} - \sqrt{f_1(R_0)}}{R_0} \right) \quad (3)$$

* habib.mazhari@emu.edu.tr

† mustafa.halilsoy@emu.edu.tr

and

$$p_0 = \frac{1}{8\pi G} \left(\frac{f'_2(R_0)}{2\sqrt{f_2(R_0)}} - \frac{f'_1(R_0)}{2\sqrt{f_1(R_0)}} + \frac{\sqrt{f_2(R_0)} - \sqrt{f_1(R_0)}}{R_0} \right) \quad (4)$$

where a prime means $\frac{d}{dR_0}$. Next, we set

$$f_1 = 1 - \frac{r_1^2}{\ell^2} \quad (5)$$

and

$$f_2 = 1 - \frac{2M}{r_2} + \frac{Q^2}{r_2^2} \quad (6)$$

as representatives of the inner (f_1) and the outer (f_2) spacetimes, respectively. Our aim is to glue the two spacetimes smoothly such that both σ_0 and p_0 vanish. For this we impose $f_1(R_0) = f_2(R_0)$ and $f'_1(R_0) = f'_2(R_0)$ which leads to

$$M = \frac{2R_0^3}{\ell^2} \quad (7)$$

and

$$Q^2 = \frac{3R_0^4}{\ell^2}. \quad (8)$$

Let us add that in this identification the dimensions of M and Q are same as R_0 and ℓ . For a double horizon case we must have the condition $R_0 > \frac{\sqrt{3}}{2}\ell$ satisfied. The choice $R_0 = \frac{\sqrt{3}}{2}\ell$ will obviously correspond to the extremal RN. It is observed that for a nontrivial matching the limit $R_0 \rightarrow 0$, must be excluded. In Fig. 1 we plot $f(r) = f_1(r)\Theta(R_0 - r) + f_2(r)\Theta(r - R_0)$ in which $\Theta(\cdot)$ stands for the Heaviside step function, for different values of Q and M (and consequently R_0 and ℓ^2).

III. STABILITY OF THE MODEL

Once we adopt that the two spacetimes are glued on the timelike shell $F := r - R = 0$ we investigate next its stability. Here we assume a radial perturbation of the shell which causes R changing with respect to the proper time τ . The standard calculation of the energy-momentum tensor of the shell when $R = R(\tau)$ yields

$$\sigma = -\frac{1}{4\pi G} \left(\frac{\sqrt{f_2(R) + \dot{R}^2} - \sqrt{f_1(R) + \dot{R}^2}}{R(\tau)} \right), \quad (9)$$

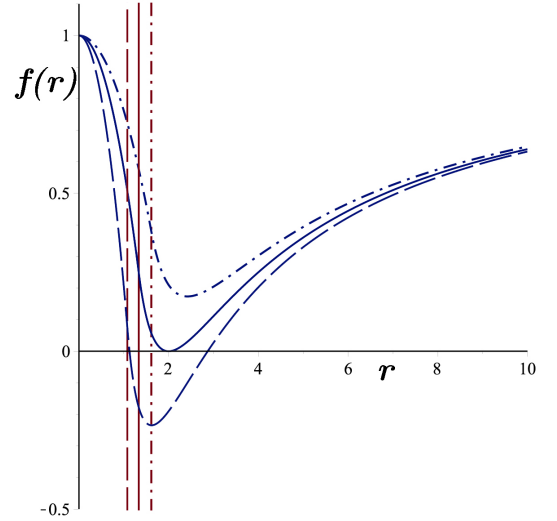


FIG. 1: The metric function $f(r) = f_1(r)\Theta(R_0 - r) + f_2(r)\Theta(r - R_0)$ versus r for different values of Q and $M = 2$. From top to bottom: $Q = 2.2, 2$ and 1.8 (or $(R_0 = 1.61, \ell^2 = 4.20)$, $(R_0 = 4/3, \ell^2 = 64/27)$ and $(R_0 = 1.08, \ell^2 = 1.26)$). The vertical lines are the locations of the interface shell, i.e. $r = R_0$.

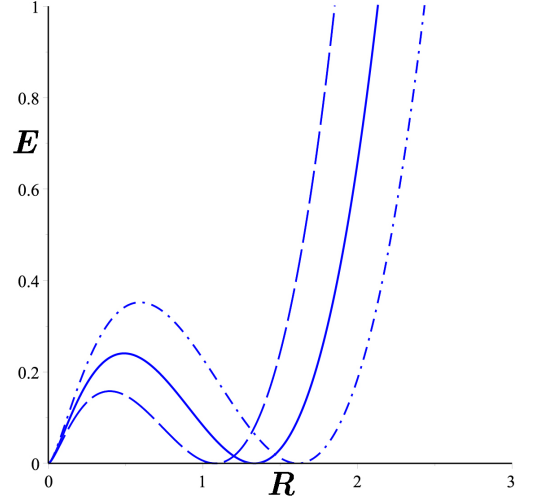


FIG. 2: E vs R for $\omega = 1$, $\nu = \frac{1}{2}$ and R_0 corresponding to Fig. 1. The energy density at $R = R_0$ is zero and so is p . Note that Dash-dot, Solid and Long-dash curves in Fig. 1 and 2 are corresponding to each other.

and

$$p = \frac{1}{8\pi G} \left(\frac{2\ddot{R}(\tau) + f'_2(R)}{2\sqrt{f_2(R) + \dot{R}^2}} - \frac{2\ddot{R}(\tau) + f'_1(R)}{2\sqrt{f_1(R) + \dot{R}^2}} + \frac{\sqrt{f_2(R) + \dot{R}^2} - \sqrt{f_1(R) + \dot{R}^2}}{R(\tau)} \right). \quad (10)$$

in which f_1 and f_2 are given in (5) and (6) and a dot represents $\frac{d}{d\tau}$. We note that the energy conservation equation imposes that σ and p given in (9) and (10) satisfy

$$\frac{d\sigma}{dR} + \frac{2}{R}(p + \sigma) = 0. \quad (11)$$

An EoS in the form of $p = p(\sigma)$ in this equation manifests the exact form of σ and p after the perturbation irrespective of the form of f_1 and f_2 . The latter equation admits

$$\int_0^\sigma \frac{d\sigma}{p(\sigma) + \sigma} = 2 \ln \left(\frac{R_0}{R} \right) \quad (12)$$

which suggests that $p(\sigma)$ can not be an arbitrary function as it must satisfy $p(0) = 0$. Note that the integration constant R_0 is identified as the unperturbed radius of the shell. For instance a linear gas with EoS $p = \omega\sigma$ ($\omega = \text{const.}$) can not be a physical choice. An equation of state of the form

$$p = -\sigma + \omega\sigma^\nu \quad (13)$$

in which $0 < \nu < 1$ is a suitable candidate for the fluid presented on the surface of the shell after the perturbation. This is a modified version of a polytropic fluid [10]. We note that the non-zero p and σ after the perturbation can be attributed to the energy given during the perturbation. Obviously it is observed from (13) that $p = 0$ when $\sigma = 0$. Integrating (12) with (13) one finds

$$\sigma(R) = \left(2\omega(1 - \nu) \ln \left(\frac{R_0}{R} \right) \right)^{\frac{1}{1-\nu}}. \quad (14)$$

Herein ω is a constant which can be adjusted but as R gets values on both sides of R_0 one has to set ν in such a way that the right-hand side remains real. For instance $\nu = \frac{1}{2}$ leaves the expression real while $\nu = \frac{3}{5}$ does not.

The total energy on the shell can be obtained as

$$E = \int \sigma(R) \delta(r - R) \sqrt{-g} d^4x = 4\pi R^2 \left(2\omega(1 - \nu) \ln \left(\frac{R_0}{R} \right) \right)^{\frac{1}{1-\nu}}. \quad (15)$$

In Fig. 2 we plot E versus R for $\omega = 1$, $\nu = \frac{1}{2}$ and for the three different R_0 values used in Fig. 1. In accordance with Fig. 2, for some extension, more deviation from $R = R_0$ requires more energy and physically this is an indication of stability. From Fig. 2 it is also seen that the well of energy formed at $R = R_0$ is strongly stable from right side. From the left, on the other hand, overcoming the energy barrier causes the shell to collapse leaving behind a flat spacetime in accordance with (7) and (8).

IV. CONCLUSION AND DISCUSSION

By applying the cut and paste technique via a thin-shell we regularize the inner part of the RN spacetime which removes its singularity. Simply the patched regular de Sitter spacetime constitutes the inner part. This amounts to the choice of the distributional metric function $f(r) = \left(1 - \frac{r^2}{\ell^2}\right) \Theta(R - r) + \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) \Theta(r - R)$, in which R stands for the radius of the shell. At the static case $R = R_0$, applying the Israel junction conditions with no matter presented on the surface for a smooth match. This requires that the metric and its first derivative are continuous on the shell which results in $M = \frac{2R_0^3}{\ell^2}$ and $Q^2 = \frac{3R_0^4}{\ell^2}$ and upon the identification of M and Q the second derivative $f''(r)$ yields no Dirac delta function. This means that the shell hosts no source, i.e. $p_0 = \sigma_0 = 0$. However, upon radial perturbation we can have $R > R_0/R < R_0$, and a source of modified polytropic fluid naturally arises. Relying on the curves of energy versus R we predict a restricted stability of the shell which makes the model feasible to certain extend. Finally, we must add that RN singularity is a time-like one which may be considered weaker than the spacelike singularity of the Schwarzschild black hole. Although our method has no immediate recipe for the removal of the latter's singularity for the RN case it finely works. We add that different method of removing the singularity of black holes is available in the literature [11].

-
- [1] G. Darmois (1927) Mémorial de Sciences Mathématiques, Fascicule XXV, "Les equations de la gravitation einsteinienne", Chapitre V.
W. Israel "Thin shells in general relativity". II. Nuovo Cim. **66**, 1 (1966).
[2] M. Visser, Phys. Rev. D **39**, 3182 (1989).

- [3] M. S. Morris and K. S. Thorne, Am. J. Phys. **56**, 395 (1988).
[4] J. Maldacena and L. Susskind, Fortsch. Phys. **61**, 781 (2013).
[5] A. Einstein and N. Rosen, Phys. Rev. **48**, 73 (1935).

- [6] A. V. Vilenkin and P. I. Fomin, Nuovo Cimento Soc. Ital. Fis., A **45**, 59 (1978).
- [7] A. V. Vilenkin and P. I. Fomin, ITP Report No. ITP-74-78R, 1974.
- [8] O. P. Zaslavskii, Phys. Rev. D **70**, 104017 (2004).
- [9] John A. Wheeler (1962) "Geometrodynamics" Acad. Press.
- [10] M. Azam, S. A. Mardan, I. Noureen and M. A. Rehman, Eur. Phys. J. C **76**, 315 (2016).
- [11] F. R. Klinkhamer and C. Rahmede, Phys. Rev. D **89**, 084064 (2014);
F.R. Klinkhamer, Acta Phys. Pol. B **45**, 5 (2014);
F.R. Klinkhamer, Mod. Phys. Lett. A **29**, 1430018 (2014);
F.R. Klinkhamer, Mod. Phys. Lett. A **28**, 1350136 (2013).